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STRESSES IN A SYMMETRICALLY LAMINATED PLATE WEAKENED BY A CENTRAL CRACK

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The axisymmetric state of stress of a piecewise-homogeneous infinite plate bonded from parallel layers and weakened by a transverse slit (crack) is considered. This problem is interesting in connection with some questions of computing the strength of rock strata. An investigation of the problem reduces to the solution of an integral equation in a function characterizing the change in the slit shape. The singularity of the solution is isolated, permitting a detailed study of the stress field including the edges and ends of the slit. Some numerical results are presented.

1. Let us consider the state of stress of a plate rigidly bonded together from strips of different elastic characteristics. The layers are assumed elastic, isotropic, and symmetric relative to the middle layer in both the elastic and geometric characteristics. The middle of the strip is slit perpendicularly to the boundary, and the plate itself is subjected to tension along the layers (Fig. 1). Let us take the following boundary conditions on the contour of the slit:

$$\sigma_x = p(y), \quad \tau_{xy} = 0, \quad x = 0, \quad |y| < 1 \qquad (1.1)$$

where $p(y)$ is an even function.

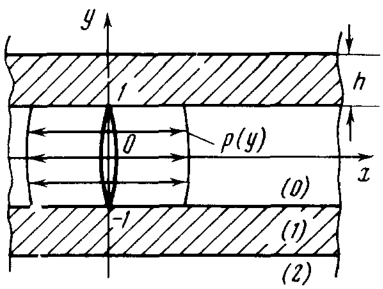

Fig. 1

The quantities referring to the middle layer (0), the layers (1) and the semi-infinite plates (2) will be denoted by the indices 0, 1, 2, respectively. Taking account of more general boundary conditions, such as addition of layers between the medium (1) and (2), is not difficult in principle and the form taken for the problem facilitates carrying out computations.

Let us represent the stress components as the sum of two terms, one of which corresponds to the stresses caused by a tension on a plate without a slit, and the other by stresses caused by the presence of the slit. The former can be determined from the condition of equality of the strains along the layers

$$C_0\overset{\circ}{\sigma}_{x0} = C_1\overset{\circ}{\sigma}_{x1} = C_2\overset{\circ}{\sigma}_{x2}, \quad \overset{\circ}{\sigma}_{yi} = 0, \quad \overset{\circ}{\tau}_{xyi} = 0 \qquad (1.2)$$

Here $C_i = (1 - \nu_i^2) / E_i$ for plane strain, and $C_i = 1 / E_i$ for the plane state of stress, E_i, ν_i are the Young's modulus and Poisson's ratio, $i = 0, 1, 2$. Because of the symmetry of the problem in both coordinates, let us consider just the lower right quadrant, for example. The additional stress components should vanish at infinity, and satisfy conditions (1.1) together with (1.2) on the slit contour.

Let $\Phi_i(x, y)$ denote the stress function. If the results of [1] are used, then after manipulation we obtain the solution for a composite semi-infinite plate $y \leqslant -1$ in the case when normal $\sigma_y = f(x)$ and shear $\tau_{xy} = g(x)$ tractions act on the boundary $y = -1$

$$\pi\Phi_1(x, y) = 2\int_0^\infty \cos\lambda x\,(f_1F_1 + f_2F_2)\frac{d\lambda}{\lambda^2 D}$$

$$\pi\Phi_2(x, y) = 2\delta\int_0^\infty e^k\cos\lambda x\,[(\varphi_2 - \varphi_1 k)F_1 + (\varphi_4 - \varphi_3 + \varphi_3 k)F_2]\frac{d\lambda}{\lambda^2 D}$$

Here

$$F_1 = \int_0^\infty f(t)\cos\lambda t\,dt, \qquad F_2 = \int_0^\infty g(t)\sin\lambda t\,dt$$

$$f_1 = [\varphi_2(a + \delta) - \varphi_1(\delta - c)]\,\mathrm{sh}\,k + \varphi_2\delta\,\mathrm{ch}\,k + (\varphi_2 a - \varphi_1 b)\,k\,\mathrm{sh}\,k - (\varphi_2 a + \varphi_1 c)\,k\,\mathrm{ch}\,k$$

$$f_2 = [\varphi_4(a + \delta) - \varphi_3 b]\,\mathrm{sh}\,k + (\varphi_4 - \varphi_3)\,\delta\,\mathrm{ch}\,k + (\varphi_4 a + \varphi_3 c)\,k\,\mathrm{sh}\,k - (\varphi_4 a - \varphi_3 b)k\,\mathrm{ch}\,k$$

$$D = -\delta^2 + (b^2 - c^2)\sigma^2 - (b^2 - c^2 + 2c\delta)\,\mathrm{sh}^2\sigma - b\delta\,\mathrm{sh}\,2\sigma$$

$$\varphi_1 = \delta(A_0 + B_0) + a(B_0 + \sigma A_0 - \sigma B_0)$$

$$\varphi_2 = \delta A_0 + bB_0 + \sigma(bA_0 + cB_0)$$

$$\varphi_3 = -\delta(A_0 + B_0) - a(B_0 - \sigma A_0 + \sigma B_0)$$

$$\varphi_4 = -\delta A_0 - bB_0 + \sigma(bA_0 + cB_0)$$

$$A_0 = \mathrm{ch}\sigma, \quad B_0 = \mathrm{sh}\,\sigma, \quad \sigma = \lambda h, \quad k = \lambda(y + h + 1)$$

$$a = \beta_1 - \beta_2 + \beta_3, \quad b = 2\beta_3, \quad c = -\beta_1 + \beta_2 + \beta_3, \quad \delta = 2\beta_2$$

for plane strain

$$\beta_1 = \beta_3\gamma_2 - \gamma_1, \quad \beta_2 = 1, \quad \beta_3 = C_2 / C_1, \quad \gamma_i = \nu_i / (1 - \nu_i)$$

and for plane stress

$$\beta_1 = \gamma_2 - \gamma_1\beta_2, \quad \beta_2 = C_1 / C_2, \quad \beta_3 = 1, \quad \gamma_i = \nu_i$$

Let us seek the solution for a half-strip as

$$\Phi_0(x, y) = L(x, y) + H(x, y)$$

$$H(x, y) = 2\int_0^\infty \cos\lambda x\,(A_1 \operatorname{ch}\lambda y + B_1\lambda y \operatorname{sh}\lambda y)\frac{d\lambda}{\pi\lambda^2} + L_1$$

$$L(x, y) = 2\int_0^\infty \cos sy e^{-sx}(A_2 + B_2 sx)\frac{ds}{\pi s^2} + L_2$$

Here $H(x, y)$ is the solution for the strip $|y| \leqslant 1$, $L(x, y)$ is the solution for the half-plane $x \geqslant 0$ and $A_1(\lambda)$, $B_1(\lambda)$, $A_2(s)$, $B_2(s)$ are arbitrary functions, L_i are arbitrary constants.

Let us introduce an unknown function on the contour

$$C_0 r(y) = \partial v / \partial x \quad \text{for } x = 0, \quad |y| < 1 \tag{1.3}$$

which is evidently determined only by the solution $L(x, y)$ and is related to a change in slit shape. Then by satisfying the second of conditions (1.1) and taking account of (1.3), we find that

$$A_2 = B_2 = \frac{1}{2}R(s), \qquad R(s) = \int_0^1 r(t)\sin st\,dt$$

The condition of equality of the stresses and displacements on the line $y = -1$ results in a system of four linear algebraic equations in the unknowns F_1, F_2, A_1, B_1, the first two of which are Fourier transforms of the unknown contact stress resultants. As a result of solving this system we obtain

$$F_j = \frac{(-1)^{j+1}D_3}{2\Delta\Delta_0}\{[(\operatorname{sh}^2\lambda + \Delta_0)a_{1j} - \lambda a_{2j}]s_1 + [(\operatorname{ch}^2\lambda + \Delta_0)a_{2j} - \lambda a_{1j}]s_2\}, \quad j = 1, 2 \tag{1.4}$$

$$A_1 = \frac{1}{4\Delta_0}[-(4F_1 + s_1)(\operatorname{sh}\lambda + \lambda\operatorname{ch}\lambda) + (4F_2 - s_2)\lambda\operatorname{sh}\lambda]$$

$$B_1 = \frac{1}{4\Delta_0}[(4F_1 + s_1)\operatorname{sh}\lambda - (4F_2 - s_2)\operatorname{ch}\lambda]$$

Here

$$s_1 = -\int_{-1}^1 r(t)\eta e^{-\eta}dt, \quad s_2 = \int_{-1}^1 r(t)(1-\eta)e^{-\eta}dt, \quad \eta = \lambda(1-t)$$

$$a_{11} = \frac{D_2 f_{22}}{\lambda^2 D} + \frac{2D_3\operatorname{ch}^2\lambda}{\Delta_0}, \qquad f_{ij} = \left.\frac{\partial^j f_i}{\partial y^j}\right|_{y=-1}$$

$$a_{12} = \frac{D_2 f_{12}}{\lambda^2 D} + \frac{2D_3\lambda}{\Delta_0} + D_1 - D_3$$

$$a_{21} = \frac{D_2 f_{23}}{\lambda^3 D} + \frac{2D_3\lambda}{\Delta_0} + D_1 - D_3 - 2D_2$$

$$a_{22} = \frac{D_2 f_{13}}{\lambda^3 D} + \frac{2D_3\operatorname{sh}^2\lambda}{\Delta_0}$$

$$\Delta = a_{22}a_{11} - a_{12}a_{21}, \quad \Delta_0 = (\operatorname{sh}2\lambda + 2\lambda)/2$$

where for plane strain

$$D_1 = D_3\gamma_0 - \gamma_1, \quad D_2 = 1, \quad D_3 = C_0 / C_1$$

and for plane stress

$$D_1 = \gamma_0 - D_2\gamma_1, \quad D_2 = C_1 / C_0, \quad D_3 = 1$$

By using the expression found for $\Phi_0(x, y)$, after a number of manipulations, the first of conditions (1.1) is reduced to the following:

$$\int_{-1}^{1} r(t)\,\omega(t, y)\,dt = -2\pi\,[p(y) - \overset{\circ}{\sigma}_{x0}], \quad |y| < 1 \tag{1.5}$$

$$\omega(t, y) = \frac{2t}{t^2 - y^2} + d_1\left[\frac{1}{2 - y - t} + \frac{1}{2 + y - t}\right] +$$

$$(3d_2 + 2)\left[\frac{1 - t}{(2 - y - t)^2} + \frac{1 - t}{(2 + y - t)^2}\right] + d_2\left[\frac{1 - y}{(2 - y - t)^2} + \frac{1 + y}{(2 + y - t)^2}\right] -$$

$$4(d_2 + 1)\left[\frac{(1 - t)(1 - y)}{(2 - y - t)^3} + \frac{(1 - t)(1 + y)}{(2 + y - t)^3}\right] - \sum_{n=0}^{\infty} H_n(y)\,t^{2n+1}$$

Here

$$d_1 = [4D_3(2b_{11} + b_{12}) / \Delta_1] - 1, \quad d_2 = -4D_3(b_{11} + b_{12}) / \Delta_1$$

$$b_{ij} = a_{ij}(\infty), \quad b_{ii} = 2(D_2 + D_3), \quad b_{ij} = D_1 + D_2 - D_3, \quad i \neq j$$

$$\Delta_1 = b_{22}b_{11} - b_{12}b_{21}$$

$$H_n(y) = \frac{D_3}{\pi(2n + 1)!}\int_0^{\infty} \frac{\lambda^{2n+1}e^{-2\lambda}}{\Delta_0}\left\{\psi_2\left[(\lambda - 2n - 1)\left(\frac{e^{-3\lambda} - 4\lambda e^{-\lambda}}{2D_3} -\right.\right.\right.$$

$$\left.h_{21}\operatorname{ch}\lambda - h_{11}\operatorname{sh}\lambda\right) + h_{21}\operatorname{ch}\lambda + \left.\frac{\lambda(a_{21} - a_{11})e^{\lambda}}{\Delta\Delta_0}\right] +$$

$$\psi_1\left[(\lambda - 2n - 2)\left(\frac{e^{-3\lambda} - 4\lambda e^{-\lambda}}{2D_3} - h_{22}\operatorname{ch}\lambda - h_{12}\operatorname{sh}\lambda\right) -\right.$$

$$\left.\left. h_{12}\operatorname{sh}\lambda + \frac{\lambda(a_{22} - a_{12})e^{\lambda}}{\Delta\Delta_0}\right]\right\}d\lambda +$$

$$\frac{8D_3}{\pi\Delta_1(2n + 1)!}\int_0^{\infty} e^{-4\lambda}\lambda^{2n+1}\left\{(\lambda - 2n - 1)\left[b_{11}(\psi_2 e^{-\lambda} - \psi_4) +\right.\right.$$

$$\left. b_{12}(\psi_1 e^{-\lambda} - \psi_3) + \frac{\Delta_1\psi_4}{4D_3}\right] - (\lambda - 2n - 2)\left[b_{21}(\psi_2 e^{-\lambda} + \psi_4) +\right.$$

$$\left.\left. b_{22}(\psi_1 e^{-\lambda} + \psi_3) - \frac{\Delta_1}{4D_3}\psi_3\right]\right\}d\lambda, \qquad \psi_1 = (2\operatorname{ch}\lambda -$$

$$\lambda\operatorname{sh}\lambda)\operatorname{ch}\lambda y + \lambda y\operatorname{ch}\lambda\operatorname{sh}\lambda y, \qquad \psi_2 = (\operatorname{sh}\lambda - \lambda\operatorname{ch}\lambda)\operatorname{ch}\lambda y +$$

$$\lambda y\operatorname{sh}\lambda\operatorname{sh}\lambda y, \qquad \psi_3 = \operatorname{ch}\lambda y + {}^1/_2\lambda(\operatorname{ch}\lambda y + y\operatorname{sh}\lambda y)$$

$$\psi_4 = -{}^1/_2\operatorname{ch}\lambda y - {}^1/_2\lambda(\operatorname{ch}\lambda y + y\operatorname{sh}\lambda y)$$

$$h_{ij} = \frac{a_{ij}}{\Delta\Delta_0}e^{2\lambda} - 16\frac{\Delta_0}{\Delta_1}e^{-2\lambda}b_{ij}$$

The problem is therefore reduced to the solution of an integral equation of the first kind in the unknown function $r(t)$.

2. Methods for the numerical solution of an equation of type (1.5) have been described in [2, 3]. Let us seek an approximate solution of the problem in the form

$$r(t) = \Gamma_0\left[\frac{(1+t)^{2-p_0}}{(1-t)^{1-p_0}} - \frac{(1-t)^{2-p_0}}{(1+t)^{1-p_0}}\right] + (1-t^2)^{1/2}\sum_{l=1}^{M}\Gamma_l U_{2l-1}(t) \qquad (2.1)$$

Here $U_{2l-1}(t)$ are odd Chebyshev functions of the second kind, $0 < p_0 < 1$. Substituting (2.1) into (1.5) and evaluating the integrals ahead of the coefficient Γ_0, we arrive at the deduction that compliance with the following condition

$$\cos\pi p_0 + 2(d_2+1)\,p_0^2 + d_1 + d_2 = 0 \qquad (2.2)$$

is necessary for the boundedness of σ_x, for $x = 0$, $|y| \leqslant 1$.

This equation is independent of E_2, ν_2. Its root $0 < p_0 < 1$ governs the nature of the growth of the stress components at the ends of the slit. Calculations confirm that such a root exists, and is moreover unique for each of the real values E_i, ν_i, $i = 0, 1$. The singularity encountered in (2.2) was first investigated in [4] in an examination of the stresses at the ends of a crack on the interface of two media.

In order to determine the unknown coefficients Γ_0, Γ_l $(l = 1, 2, \ldots M)$, let us write (1.5) at the collocation nodes y_l and we obtain a system of linear algebraic equations. Roots of the Chebyshev polynomials of the first kind

$$y_l = \cos\frac{(2l-1)\pi}{4(M+1)}, \qquad l = 1, 2, \ldots, (M+1) \qquad (2.3)$$

are selected as collocation nodes. Here $M+1$ is the number of points dividing the segment [0, 1]. It is interesting to note that $A_1 = B_1 = 0$ for a homogeneous plate $(E_i = E,\ \nu_i = \nu,\ p_0 = 0.5)$; if $p(y) = 0$, $\sigma_{x0}^{\circ} = 1$, then $\Gamma_0 = 0.5$, $\Gamma_l = 0$ and the stress function $L(x, y)$ yields quadratures, completely in agreement with the known solution [5] for an infinite plane with a rectilinear slit, for the stress components. Therefore, the series in (2.1) is a correction due to the inhomogeneity of the plate.

The computation was carried out for $M = 11$. The integrals $H_n(y)$ were evaluated by Simpson's rule. The remaining integrals in (1.5M) are expressed in quadratures or are represented by hypergeometric series.

The case of plane strain is considered in the examples presented below. It is moreover assumed that $\sigma_{x_0}^{\circ} = 0$, $p(y) = -1$, $\nu_0 = \nu_1 = \nu_2 = 0.25$.

Values of the contour stresses σ_y are given in Table 1 for $E_2 / E_0 = 1$, $h = 1$ and for different values of the ratio E_1 / E_0 (indicated in the upper row); $\sigma_y = -1$ for $E_1 / E_0 = 1$.

Table 1

y	3	5	10	100	1000
0.999	7.440	8.543	8.979	8.931	8.855
0.95	0.325	0.675	0.967	1.262	1.281
0.9	—0.156	0.086	0.298	0.533	0.550
0.8	—0.521	—0.372	—0.233	—0.062	—0.049
0.6	—0.780	—0.702	—0.623	—0.510	—0.501
0.4	—0.879	—0.830	—0.776	—0.689	—0.682
0.2	—0.928	—0.887	—0.844	—0.768	—0.763
0.0	—0.936	—0.903	—0.864	—0.792	—0.787

Setting $E_2 = 0$, $E_1 = E_0$, we obtain a strip with centered lateral slit. Presented in Table 2 are values $(-\sigma_y)$ on the edge for different values of the ratio between the slit length and the width of the strip $\varepsilon = (1 + h)^{-1}$ (indicated in the upper row).

Table 2

y	0.8	0.6	0.4	0.2	0.1
0.999	2.454	1.423	1.130	1.026	1.007
0.95	2.365	1.409	1.127	1.026	1.006
0.9	2.268	1.394	1.126	1.026	1.006
0.8	2.085	1.367	1.122	1.026	1.006
0.6	1.805	1.321	1.116	1.025	1.006
0.4	1.635	1.289	1.111	1.025	1.006
0.2	1.544	1.270	1.108	1.025	1.006
0.0	1.516	1.264	1.107	1.025	1.006

The solution of this problem by reduction to a Fredholm integral equation of the second kind has been presented in [6], where the coefficient of stress intensity is determined by the formula (l is half the crack length)

$$N = -\frac{E}{2} l^{1/2} (1 - \nu^2)^{-1} \lim_{y \to 1} [v_x'(x, y)(1 - y)^{1/2}]$$

In the case under consideration here

$$N = -\sqrt{2}\Gamma_0$$

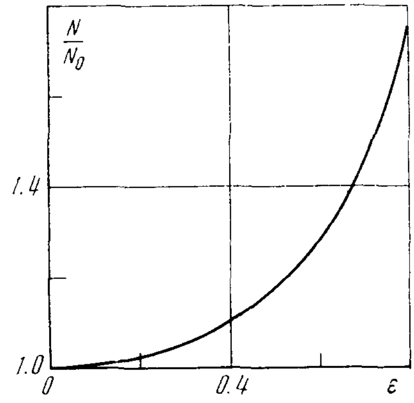

Fig. 2

A comparison of the results yields an error not exceeding 0.65% for $\varepsilon \leqslant 0.5$. The deduction can be made that the intensity coefficients and the stress components for $\lambda \leqslant 0.2$ do not differ, in practice, from those for an infinite plate with a slit. The growth of N / N_0 as a function of ε is shown in Fig. 2. Here $N_0 = N|_{\varepsilon=0}$. Computations show the high degree of convergence of the method in the case $E_1 = E_0$. For $E_1 \neq E_0$ the convergence of the solution is somewhat worse but it is almost independent of the values of the ratio between the layer stiffnesses.

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